There is no stationary cyclically monotone Poisson matching in 2D

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The random Euclidean bipartite matching problem

Let X_i, Y_i be i. i. d. uniformly distributed points $(X_i)_{i=1}^n, (Y_i)_{i=1}^n$ on $(0,1)^d$ and consider the optimization problem

$$\min_{\sigma \in S_n} \sum_{i=1}^{n} \left| X_i - Y_{\sigma(i)} \right|^\gamma .$$

Define the empirical measures

$$\mu_n = \sum_{i=1}^{n} \delta_{X_i}, \quad \nu_n = \sum_{i=1}^{n} \delta_{Y_i}.$$

Recall the Wasserstein distance:

$$W_\gamma^\gamma (\mu_n, \nu_n) = \inf_{q \in \mathsf{Cpl}} \int |x-y|^\gamma \, dq(x,y),$$

where Cpl is the set of coupling between μ_n and ν_n, namely the set of measures on the product space with first marginal equals to μ_n and second marginal equals to ν_n.

By Birkhoff's theorem

$$W_\gamma^\gamma(\mu_n, \nu_n) = \min_{\sigma \in S_n} \sum_{i=1}^{n} \left| X_i - Y_{\sigma(i)} \right|^\gamma .$$

Macroscopic Behaviour

Idea: The typical distance is $n^{-\frac{1}{d}}$ (points are spread as in a regular grid)

$$\min_{\sigma \in S_n} \sum_{i=1}^{n} \left| X_i - Y_{\sigma(i)} \right|^{\gamma} \approx n \cdot n^{-\frac{\gamma}{d}}.$$

Achtung: There are fluctuations! (CLT)

The asymptotic behaviour of $W_\gamma(\mu_n, \nu_n)$ depends on the dimension d and for $\gamma \geq 1$ read as follows:

$$W_\gamma^\gamma(\mu_n, \nu_n) \sim \begin{cases} n \cdot n^{-\frac{\gamma}{2}} & \text{for } d = 1, \\ n \cdot \left(\frac{\ln n}{n}\right)^{\frac{\gamma}{2}} & \text{for } d = 2, \\ n \cdot n^{-\frac{\gamma}{d}} & \text{for } d \geq 3. \end{cases}$$

The critical dimension $d = 2$ has been firstly understood in the seminal paper by Ajtai, Komlós and Tusnády (1983).

Thermodynamic limit: Consider $[-\frac{L}{2}, \frac{L}{2}]^d$ or $\mathbb{T}_L^d$ and throw L^d point into them, so that the typical interpoint distance is 1. And let q_L be the γ-minimal coupling.
$\rightsquigarrow$ Let $L \to \infty$.

Question: Which property should the limiting coupling q_∞ satisfy?

The Poisson Point Process

The Poisson point process on $\mathbb{R}^d$ can be defined as a random variable taking values on locally finite atomic measures

$$\mu = \sum_i \delta_{X_i}$$

such that for every $k \geq 1$, for any disjoint Borel sets $A_1, \ldots, A_k \subseteq \mathbb{R}^d$,

- $\rightsquigarrow$ the random variables $\mu(A_1), \ldots, \mu(A_k)$ are independent,
- $\rightsquigarrow$ the random variable $\mu(A_i)$ has a Poisson distribution of parameter $|A_i|$ for every $i = 1, \ldots, k$.

Existence: Superposition argument. On $\Omega \subseteq \mathbb{R}^d$ bounded, conditionally on $\mu(\Omega) = n$, the measure $\mu \llcorner \Omega$ has the same law as the random measure

$$\sum_{i=1}^{n} \delta_{X_i},$$

where $(X_i)_{i=1}^n$ are i. i. d. with uniform law on Ω.

Notation: I will often make use of the notation $\{X\}$ to denote a Poisson point process.

Motivation

Definition: Consider two Poisson point process $\{X\}, \{Y\}$ in $\mathbb{R}^d$ and let T be a bijection from $\{X\}$ to $\{Y\}$. We call matching the triple $(\{X\}, \{Y\}, T)$.

Definition: A matching $(\{X\}, \{Y\}, T)$ is γ-minimal if for any finite subset $\{X_i\}_{i=1}^n \subset \{X\}, \{Y_i\}_{i=1}^n = \{T(X_i)\}_{i=1}^n \subset \{Y\}$

$$\sum_{i=1}^{n} |T(X_i) - X_i|^\gamma = \min_{\sigma \in S_n} \sum_{i=1}^{n} \left|X_i - Y_{\sigma(i)}\right|^\gamma .$$

Definition: a matching $(\{X\}, \{Y\}, T)$ is said to be stationary if the joint law of $(\{X\}, \{Y\}, T)$ is invariant under the action of the additive group $\mathbb{Z}^d$

$$(\{X\}, \{Y\}, T) \mapsto (\{x + X\}, \{x + Y\}, T(\cdot - x) + x) \quad x \in \mathbb{Z}^d.$$

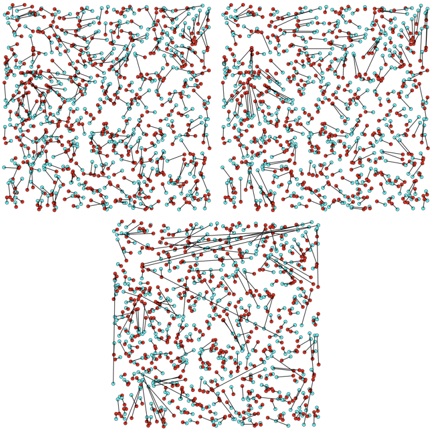
Question: (Peres 2002) For $\{X\}, \{Y\}$ independent Poisson processes of intensity 1 in $\mathbb{R}^2$, does there exist a stationary *planar* matching?

Question: (Holroyd 2009) For $\{X\}, \{Y\}$ independent Poisson processes of intensity 1 in $\mathbb{R}^2$, does there exist a stationary γ-*minimal* matching?

Motivation

Achtung: Infinite cost does not imply non existence.
Let $d = 1, \gamma = \frac{1}{2}$ and let $(X_i)_{i=1}^L, (Y_i)_{i=1}^L$ i. i. d. uniformly distributed on $[-\frac{L}{2}, \frac{L}{2}]$

$$\min_{\sigma \in S_n} \sum_{i=1}^{L} \left|X_i - Y_{\sigma(i)}\right|^{\frac{1}{2}} \sim \ln L.$$



γ-minimal matchings for $\gamma = \infty$ (top-left), $\gamma = 1$ (top-right), and $\gamma = -\infty$ (bottom).
Credits to Holroyd-Janson-Wästlund 2020.

Theorem (Holroyd-Janson-Wästlund 2020)

There exists a stationary γ-minimal matching if

- $\rightsquigarrow$ $d = 1, \gamma < 1$;
- $\rightsquigarrow$ $d = 2, \gamma < 1$;
- $\rightsquigarrow$ $d \geq 3, \gamma < \infty$.

Our result

We are interested in the case $\gamma = 2$, $d = 2$.

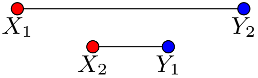
Theorem (Huesmann-M.-Otto)

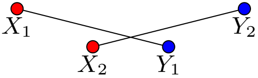
There exists no stationary, ergodic and 2*-minimal matching* $(\{X\}, \{Y\}, T)$ *in* $d = 2$.

Idea of the proof: Contradiction argument. We argue by showing that stationarity together with 2-minimality imply the following contradiction:

$$O(\ln^{\frac{1}{2}} R) \leq \frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X| \leq o(\ln^{\frac{1}{2}} R).$$

Achtung: 2-*minimality* does not imply *planarity*. Let $X_1 = (0, c)$, $X_2 = (1, 0)$, $Y_1 = (2, 0)$, $Y_2 = (3, c)$ then





cost of parallel matching $= 10$ cost of crossing matching $= 2(4 + c^2)$

Proof in a nutshell

Step 1: Ergodic estimate

$$\#\left\{X \in (-R,R)^d \ : \ \ |T(X) - X| \gg 1\right\} \leq o(R^d).$$

Step 2: L^∞-estimate

$$|T(X) - X| \leq o(R) \text{ provided that } X \in (-R,R)^d,$$

Step 3: Harmonic approximation

$$\frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X|^2 \leq O(\ln R).$$

Step 4: Trading integrability against asymptotics.

$$\frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X| \leq o(\ln^{\frac{1}{2}} R).$$

Step 5: Lower bound

$$\frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X| \geq O(\ln^{\frac{1}{2}} R).$$

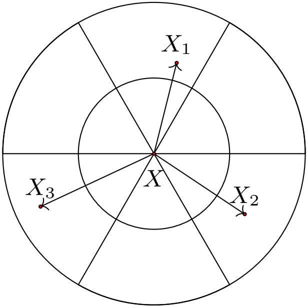
Step 2: L^∞-estimate

Lemma

For every $\varepsilon > 0$ there exists a random radius $r_* < \infty$ a. s. such that for every $R \geq r_*$

$$|T(X) - X| \leq \varepsilon R \quad \textit{provided that } X \in (-R, R)^2.$$

Idea: There are enough "good" points around X.



$$(T(X) - X) \cdot \frac{(X_i - X)}{|X_i - X|} \lesssim \frac{|T(X_i) - X_i|^2}{|X_i - X|} + |X_i - X| \lesssim \varepsilon R.$$

Step 3: Harmonic approximation

Aim: Improve the L^∞-estimate to an L^2-estimate of the local energy of the type $E(R) \leq O(\ln R)$.

Lemma

There exist a constant C and a random radius $r_* < \infty$ a. s. such that for every $R \geq r_*$ we have

$$E(R) = \frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X|^2 \leq C \ln R.$$

What we know: From Step 2 we know that $E(R) \leq \varepsilon R^2$.

Harmonic Approximation Theorem

Define the local energy

$$E(R) := \frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X|^2 .$$

Call $\mu = \sum_{X \in (-R,R)^2} \delta_X$ and $\nu = \sum_{Y \in (-R,R)^2} \delta_Y$ define the data term

$$D(R) := \frac{1}{R^d} W^2_{(-R,R)^2}(\mu, n_\mu) + \frac{R^2}{n_\mu}(n_\mu - 1)^2 + \frac{1}{R^d} W^2_{(-R,R)^2}(\nu, n_\nu) + \frac{R^2}{n_\nu}(n_\nu - 1)^2,$$

where $n_\mu = \frac{\#\{X \in (-R,R)^2\}}{4R^2}$, $n_\nu = \frac{\#\{Y \in (-R,R)^2\}}{4R^2}$.

Theorem (Goldman-Huesmann-Otto)

For any $0 < \tau \ll 1$, there exist an $\varepsilon := \varepsilon(\tau) > 0$ and a $C_\tau < \infty$ such that provided for some R

$$\frac{1}{R^2} E(6R) + \frac{1}{R^2} D(6R) \leq \varepsilon$$

there exists a harmonic gradient field Φ such that

$$\frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X - \nabla\Phi(X)|^2 \leq \tau E(6R) + C_\tau D(6R),$$

$$\sup_{B_{2R}} |\nabla\Phi|^2 \leq C_\tau \left(E(6R) + D(6R)\right).$$

Application of the harmonic approximation

Idea: Splitting the sum.

Consider the contribution given by the points which are transported by large distance

$$
\begin{aligned}
&\frac{1}{R^d}\sum_{(X\in B_R \text{ or } T(X)\in B_R) \text{ and } |T(X)-X|>L_\tau} |T(X)-X|^2 \\
&\le \frac{2}{R^d}\sum_{X\in B_R \text{ or } T(X)\in B_R} |T(X)-X-\nabla\Phi(X)|^2 \\
&+ \frac{2}{R^d}\sum_{(X\in B_R \text{ or } T(X)\in B_R) \text{ and } |T(X)-X|>L_\tau} |\nabla\Phi(X)|^2 \\
&\le 2\tau(1+C_\tau)E(6R)+2C_\tau(1+\tau)\underbrace{D(6R)}_{\lesssim \ln R}.
\end{aligned}
$$

This combines to

$$E(R)\le \tau E(6R)+C_\tau \ln R.$$

Iteration:

$$E(R)\le \tau^k E(6^k R)+C_\tau\sum_{l=0}^{k-1}\tau^l \ln R \le \varepsilon(36\tau)^k R^2 + C_\tau \sum_{l=0}^{k-1}\tau^l \ln R.$$

Step 4: Upper bound

Lemma

For every $\varepsilon > 0$ *there exists a random radius* $r_* < \infty$ *a. s. such that*

$$\frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X| \leq \varepsilon \ln^{\frac{1}{2}} R.$$

Proof: We split again the sum into moderate and large transportation distance and apply Cauchy-Schwarz:

$$\begin{aligned}
&\frac{1}{R^d} \sum_{X \in B_R \text{ or } T(X) \in B_R} |T(X) - X| \\
&\leq \frac{1}{R^d} \sum_{(X \in B_R \text{ or } T(X) \in B_R) \text{ and } |T(X) - X| \leq L} |T(X) - X| \\
&+ \frac{1}{R^d} \sum_{(X \in B_R \text{ or } T(X) \in B_R) \text{ and } |T(X) - X| > L} |T(X) - X| \\
&\leq CL + \varepsilon^{\frac{1}{2}} E(R)^{\frac{1}{2}} \leq C\varepsilon^{\frac{1}{2}} \ln^{\frac{1}{2}} R.
\end{aligned}$$

Conclusions

For $\gamma = 2$, $d = 2$ we proved the following:

Theorem

There exists no stationary, ergodic and 2-minimal matching $(\{X\}, \{Y\}, T)$ in $d = 2$.

Our proof relies on the Harmonic Approximation Theorem that requires $\gamma = 2$.

Question: What if $\gamma \geq 1$?
$\rightsquigarrow$ Next step: there exists no stationary γ-minimal coupling if $\gamma > 1$.
$\rightsquigarrow$ Problem: $\gamma = 1$ requires a different argument.

Thank you for the attention!